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LETTER TO THE EDITOR

Quantum deformations of $SU(3)$ and subalgebras

Feng Pan

Department of Physics, Liaoning Normal University, Dalian 116022, People's Republic of China (mailing address), Department of Physics, Nanjing University, Nanjing 210008, People's Republic of China and
High Energy Section, ICTP, PO Box 586, 34100 Trieste, Italy

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Abstract. Quantum deformations of $SU(3)$ is discussed. Special attention is given to $SO_q(3)$ and $SO(3)$ subalgebras of $SU_q(3)$ and $SU(3)$. Various chains of $SU_q(3)$ and $SU(3)$ are realized by using the deforming functionals. Both boson and q-boson realizations are also discussed.

Quantum algebras [1-3], or quantized universal enveloping algebras, are recently attracting much attention in both physics and mathematics. It has been shown that rotational spectra of nuclei and molecules can be described very accurately in terms of the quantum algebra $SU_q(2)$ [4-5]. The deformation parameter τ^2 (with $q=e^{i\tau}$) of the $SU_q(2)$ model for nuclei has been found [4] to correspond to the softness parameter of the VMI model, thus indicating that the q-deformation of the usual $SU(2)$ algebra is physically well motivated. In a recent letter [6], Bonatsos *et al* constructed the q-deformed version of a two-dimensional toy interacting boson model (IBM) with the symmetry $SU_q(3)\supset SU_q(2)\supset SO(2)$, which gives several hints about the possible q-generalization of the full IBM and its possible usefulness. However, in further applications of quantum algebras to nuclear or molecular physics, various algebra chains need to be constructed, e.g. in the two-dimensional toy IBM [7] in addition to the q-deformed $SU(2)$ chain of subalgebras [6] the $SU_q(3)\supset SO_q(3)\supset SO(2)$ chain should also be constructed. In the present letter we will discuss this problem. Further extensions to other quantum algebras or Lie algebras are straightforward.

It is well known that the generators of $U(3)$ can be expressed in terms of $E_{ij}\{1\leqslant i, j\leqslant 3\}$ which satisfy the Hermiticity condition

$$(E_{ij})^{\dagger}=E_{ji} \tag{1}$$

and the commutation relations

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}. \tag{2}$$

In addition to $SU(2)\times U(1)$ subalgebra chain, $U(3)$ can also be reduced to $SO(3)$ with three generators

$$L_+=E_{13}+E_{32} \qquad L_-=(L_+)^{\dagger} \qquad L_0=E_{11}-E_{22} \tag{3}$$

which satisfy the commutation relations

$$[L_0, L_\pm]=\pm L_\pm \qquad [L_+, L_-]=L_0. \tag{4}$$

It is interesting to note that the generators $E^q_{ij}\{1\leqslant i,j\leqslant 3\}$ of quantum algebra $SU_q(3)$ for symmetric irreps can be realized in terms of $U(3)$ generators by using the following deforming functionals, namely

$$\begin{aligned} E^q_{ij} &= a^+_{qi}a_{qj} = F_{ij}(g) = ([E_{ii}]/E_{ii})^{1/2}E_{ij}([E_{jj}]/E_{jj})^{1/2} \qquad \text{for } i\neq j \\ E^q_{ii} &= F_{ii}(g) = E_{ii} \end{aligned} \tag{5}$$

where for given x,

$$[x] = (q^x - q^{-x})/(q-q^{-1}) \tag{6}$$

and

$$E_{ij} = a^+_i a_j \tag{7}$$

a^+_i, a_i $(i=1,2,3)$ are boson operators, while a^+_{qi}, a_{qi} are q-boson operators. We always assume that the operators given by (5) are acting on representations in which E_{ii} $(i=1,2,3)$ are diagonal simultaneously. By dint of the commutation relations of $U(3)$, the functionals given by (5) indeed satisfy the commutation relations and Serre relations of $U_q(3)$. The maps $F_{ij}(g)$ are functionals of $U(3)$ generators $g: E_{ij}$. If one wants to keep the Hermiticity condition for E^q_{ij} for $i\neq j$, the deformation parameter q should be real or a phase $(q=\mathrm{e}^{\mathrm{i}\tau})$ with $|\tau|N<\pi$, where $N=E_{11}+E_{22}+E_{33}$. Consequently, the functionals (5) are invertible, and their inverse F^{-1} provide a realization of $SU(3)$ in terms of quantum algebra $SU_q(3)$ generators. The functionals (5) are nothing but the generalization of the deforming functionals for the q-Heisenberg algebra given by [8–10]. Deforming functionals for $SU_q(2)$ were made by Curtright and Zachos [11]. In the $SU_q(2)$ case, one first rewrites the classical Casimir operator C_2 of $SU(2)$ as $j(j+1)$, where j is the formal operator $((1+4C_2)^{1/2}-1)/2$. Then, by dint of the commutation relations of $SU(2)$ generators $j_\pm$, and j_0, the functionals

$$\begin{aligned} J^q_0 &= Q_0(j_0) = j_0 \\ J^q_+ &= Q_+(g) = ([j_0+j][j_0-j-1]/(j_0+j)(j_0-j-1))^{1/2}j_+ \\ J^q_- &= Q_-(g) = j_-([j_0+j][j_0-j-1]/(j_0+j)(j_0-j-1))^{1/2} \end{aligned} \tag{8}$$

where the operators of (8) are acting on representations where j and j_0 are diagonal simultaneously, satisfy the commutation relations of $SU_q(2)$. In this case, in order to keep the operators $J^q_\pm$ to be Hermitian, q should be real or a phase $(q=\mathrm{e}^{\mathrm{i}\tau})$ with $|\tau|(2j+1)<\pi$, otherwise the conjugation is not Hermitian.

Using the deforming functionals, we can establish the following algebra chains.

(1) $SU(3)\supset SO_q(3)\supset SO(2)$. In this case, the generators of $SU(3)$ can be expressed in terms of E_{ij} with $1\leqslant i,j\leqslant 3$. The $SO(3)$ generators can be realized by using the following functionals, namely

$$\begin{aligned} L^q_0 &= E_{11}-E_{22} \\ L^q_+ &= ([L_0+l][L_0-l-1]/(L_0+l)(L_0-l-1))^{1/2}(E_{13}+E_{32}) \\ L^q_- &= (E_{31}+E_{23})([L_0+l][L_0-l-1]/(L_0+l)(L_0-l-1))^{1/2} \end{aligned} \tag{9}$$

where

$$l = ((1+4C_2)^{1/2}-1)/2 \tag{10a}$$

with

$$C_2 = 2L_-L_+ + L_0(L_0+1) \tag{10b}$$

where $L_\pm$, L_0 are defined in equation (3). The operators of (9) are acting on representations where L_0 and C_2 are diagonal. In order to keep the Hermiticity condition for $L^q_\pm$, one should take q to be real or a phase ($q=e^{i\tau}$) with $|\tau|(2N+1)<\pi$, where

$$N=E_{11}+E_{22}+E_{33}. \tag{11}$$

Using commutation relations of $L_\pm$, L_0 given by (4), one can check that $L^q_\pm$, and L^q_0 satisfy the commutation relations of $SO_q(3)$

$$[L^q_0, L^q_\pm]=\pm L^q_\pm \qquad [L^q_+, L^q_-]=\tfrac{1}{2}[2L^q_0]. \tag{12}$$

This realization is valid for generic irreps of $SU(3)$.

(2) $U_q(3)\supset SO(3)\supset SO(2)$. In this case the generators E^q_{ij} with $1\leqslant i, j\leqslant 3$ of $U_q(3)$ satisfy

$$\begin{aligned}&[E^q_{ij}, E^q_{ii}]=-E^q_{ij} \qquad [E^q_{ij}, E^q_{jj}]=E^q_{ij}\\ &[E^q_{ij}, E^q_{ji}]=[E^q_{ii}-E^q_{jj}]\end{aligned} \tag{13a}$$

for $i\neq j$, and

$$E^q_{12}E^q_{23}-E^q_{23}E^q_{12}q^{-1}=E^q_{13}q^{E^q_{22}} \tag{13b}$$

etc. The $SO(3)$ generators can be expressed as the following deforming functionals

$$\begin{aligned}L_0&=E^q_{11}-E^q_{22}\\ L_+&=(E^q_{11}/[E^q_{11}])^{1/2}E^q_{13}(E^q_{33}/[E^q_{33}])^{1/2}+(E^q_{33}/[E^q_{33}])^{1/2}E^q_{32}(E^q_{22}/[E^q_{22}])^{1/2}\\ L_-&=(E^q_{33}/[E^q_{33}])^{1/2}E^q_{31}(E^q_{11}/[E^q_{11}])^{1/2}+(E^q_{22}/[E^q_{22}])^{1/2}E^q_{23}(E^q_{33}/[E^q_{33}])^{1/2}.\end{aligned} \tag{14}$$

In this case the operators $L_\pm$ are Hermitian only when q is real or a phase ($q=e^{i\tau}$) with $|\tau|N<\pi$, where

$$N=E^q_{11}+E^q_{22}+E^q_{33}. \tag{15}$$

This realization is only valid for symmetric irreps of $U_q(3)$ because equation (5) holds for symmetric irreps.

Generally, an operator appearing in denominator and under square root is not allowed in an abstract algebra because it is not well defined, but one can allow this provided one assumes that these operators are acting on representations in which they are diagonal. For example, the inverse expression of equation (5) is well defined in the $U_q(3)\supset SU_q(2)\times U(1)$ basis [12]

$$\left|\begin{matrix}n & 0 & 0\\ & j & m\end{matrix}\right\rangle_q=\frac{a^{+j+m}_{q1}a^{+j-m}_{q2}a^{+n-2j}_{q3}}{([n-2j]![j+m]![j-m]!)^{1/2}}|0\rangle_q. \tag{16}$$

The operators $(E^q_{ii}/[E^q_{ii}])^{1/2}$ acting on (16) gives

$$(E^q_{ii}/[E^q_{ii}])^{1/2}\frac{a^{+n_1}_{q1}a^{+n_2}_{q2}a^{+n_3}_{q3}}{([n_1]![n_2]![n_3]!)^{1/2}}|0\rangle_q=(n_i/[n_i])^{1/2}\frac{a^{+n_1}_{q1}a^{+n_2}_{q2}a^{+n_3}_{q3}}{([n_1]![n_2]![n_3]!)^{1/2}}|0\rangle_q. \tag{17}$$

It can be proved that the basis vectors of $U_q(3)\supset SO(3)\supset SO(2)$ can be expanded in terms of (16). Because the operator $(E^q_{ii}/[E^q_{ii}])^{1/2}$ acting on (16) is well defined, the deforming functionals given by (14) are also well defined when they act on the basis vectors of $U_q(3)\supset SO(3)\supset SO(2)$. In this case one can obtain the matrix elements of $(E^q_{ii}/[E^q_{ii}])^{1/2}$ though E^q_{ii} are not diagonal.

One can check that $L_\pm$, and L_0 satisfy equation (4).

(3) $U_q(3) \supset SO_{q'}(3) \supset SO(2)$. In this case the $U_q(3)$ generators are E^q_{ij} with $1 \le i, j \le 3$. The generators of $SO(3)$ subalgebra are given by (14). Using the functionals (8) and generators $L_\pm$, L_0 given by (14), the generators of $SO_{q'}(3)$ can be expressed as

$$\begin{aligned} L^{q'}_+ &= ([L^{q'}_0 + l]_{q'}[L^{q'}_0 - l - 1]_{q'}/(L^{q'}_0 + l)(L^{q'}_0 - l - 1))^{1/2} L_+ \\ L^{q'}_0 &= E^q_{11} - E^q_{22} \\ L^{q'}_- &= L_-([L^{q'}_0 + l]_{q'}[L^{q'}_0 - l - 1]_{q'}/(L^{q'}_0 + l)(L^{q'}_0 - l - 1))^{1/2} \end{aligned} \tag{18}$$

where the operators are acting on the representations in which $L^{q'}_0$ and l are diagonal, and the operator l is given by equations (10*a*) and (10*b*). In this case the operators $L^{q'}_\pm$ are Hermitian only when q is real or a phase ($q = e^{i\tau}$) with $|\tau| N < \pi$, and q' is real or a phase ($q' = e^{i\tau'}$) with $|\tau'|(2N+1) < \pi$, where N is given by (15). This realization is also valid for symmetric irreps of $U_q(3)$ due to the fact that equation (5) holds for symmetric irreps only.

In the following, we will discuss boson and q-boson realizations. In case (1), the generators of $U(3)$ can be realized by using boson operators a^+_i, a_i for $i = 1, 2, 3$ with $E_{ij} = a^+_i a_j$. Then, the highest weight state of $U(3) \supset SO_q(3) \supset SO(2)$ can be written as

$$\left| \begin{matrix} (n & 0 & 0) \\ L^q = M = n \end{matrix} \right\rangle = \frac{1}{(n!)^{1/2}} a^{+n}_1 |0\rangle \tag{19}$$

where $|0\rangle$ is boson vacuum state. In order to construct the general basis vectors, we need the following lemma.

Lemma. Let g be a classical Lie algebra and g^q be the q-deformed quantum algebra of g. If θ is a g invariant, θ must be a g^q invariant.

The validity of the lemma is evident. Using deforming functionals, one knows that any generator of g^q can be expressed in terms of generators of g. If θ commutes with g, θ must commute with g^q. However, the inverse of the lemma, generally, is not valid.

Using a^+_i ($i = 1, 2, 3$), we can construct the following $SO(3)$ invariant

$$\theta = (a^{+2}_3 - 2a^+_1 a^+_2)^k. \tag{20}$$

Using the lemma, one knows θ is also an $SO_q(3)$ invariant. Thus basis vectors with $U(3)$ irrep (n 0 0) and q-deformed angular momentum quantum number $L^q = M = L$ can be written as

$$\left| \begin{matrix} (n & 0 & 0) \\ L^q = M = L \end{matrix} \right\rangle = \left(\frac{(2L+1)!!}{(n+L+1)!!(n-L)!!L!} \right)^{1/2} (a^{+2}_3 - 2a^+_1 a^+_2)^{(n-L)/2} a^{+L}_1 |0\rangle. \tag{21}$$

Finally, the general basis vectors for symmetric irrep of $U(3)$ in the $SO_q(3)$ basis can be expressed as

$$\begin{aligned} \left| \begin{matrix} (n & 0 & 0) \\ L^q & M \end{matrix} \right\rangle &= \left(\frac{2^{L^q - M}(2L^q+1)!![L^q+M]!}{(n+L^q+1)!!(n-L^q)!!L^q![2L^q]![L^q-M]!} \right)^{1/2} \\ &\quad \times (L^q_-)^{L^q - M} (a^{+2}_3 - 2a^+_1 a^+_2)^{(n-L^q)/2} a^{+L^q}_1 |0\rangle. \end{aligned} \tag{22}$$

The generators of $U(3)$ can also be realized by using q-boson operators a^+_{qi}, a_{qi}, and N_i for $i = 1, 2, 3$ owing to the fact that the boson operators a^+_i, a_i can be expressed in terms of them, namely [8–10]

$$\begin{aligned} a^+_i a_i &= N_i \\ a^+_i &= (N_i/[N_i])^{1/2} a^+_{qi} \\ a_i &= a_{qi}(N_i/[N_i])^{1/2}. \end{aligned} \tag{23}$$

In this case

$$a_i^+ = (a_i)^\dagger \tag{24}$$

only when q is real or a phase ($q = e^{i\tau}$) with $|\tau| < \pi/N$, where N is the total number of bosons or q-bosons.

Similarly, the q-boson realizations of $U_q(3) \supset SO_{q'}(3) \supset SO(2)$ basis vectors can be expressed as

$$\left|\begin{matrix}(n & 0 & 0) \\ L & M & \end{matrix}\right\rangle_{qq'} = \left(\frac{(2L+1)!!2^{L-M}[L+M]_{q'}!}{(n+L+1)!!(n-L)!![L]![2L]_{q'}![L-M]_{q'}!}\right)^{1/2}$$
$$\times (L_-^{q'})^{L-M}\left(\left(\frac{N_3(N_3+1)}{[N_3][N_3+1]}\right)^{1/2} a_{q3}^{+2} - 2\left(\frac{N_1N_2}{[N_1][N_2]}\right)^{1/2} a_{q1}^+ a_{q2}^+\right)^{(n-L)/2}$$
$$\times a_{q1}^{+L}|0\rangle_q. \tag{25}$$

When $q' \to 1$, (25) gives the basis vectors of $U_q(3) \supset SO(3) \supset SO(2)$.

Now we turn to the coproduct definition for the above realizations. In the following, we always assume that q is not a root of unity.

For functional realization of $U_q(3)$ algebra given by (5) there are two definitions of coproduct. One is the definition originated from Hopf algebra [2]

$$\begin{aligned} \Delta(X) &= q^{(E^q_{ii} - E^q_{i+1i+1})/2} \otimes X + X \otimes q^{-(E^q_{ii} - E^q_{i+1i+1})/2} \\ \Delta(E^q_{ii}) &= 1 \otimes E^q_{ii} + E^q_{ii} \otimes 1 \end{aligned} \tag{26}$$

where X is E^q_{ii+1} or E^q_{i+1i}.

Another definition is the so-called map-induced coproduct [13]. Let g be generator of $U(3)$, and G be $SU_q(3)$. The tensor product of g is

$$\Delta(g) = 1 \otimes g + g \otimes 1. \tag{27}$$

Thus the map F from $U(3)$ generators g to $SU_q(3)$ generators G induces the following tensor product of G's

$$\Delta(G) = F(\Delta(g)) \tag{28}$$

which obeys $SU_q(3)$ commutation relations and Serre relations, since its arguments obey $U(3)$. This induced coproduct is an equivalent one. However, because the deforming functionals (5) are only valid for symmetric irreps, the corepresentation resulted from (27) should remain symmetric, otherwise the resulting corepresentations obtained from (28) will not be representations of $SU_q(3)$ anymore.

Thus, in case (1) the coproduct of $SO_q(3)$ is defined by the map-induced definition

$$\Delta(L^q) = Q(\Delta(E_{ij})) \tag{29}$$

where $\Delta(E_{ij}) = 1 \otimes E_{ij} + E_{ji} \otimes 1$ is the tensor product of $U(3)$. This definition is valid for generic irreps of $U(3)$. In cases (2), and (3) the definition of $SO_{q'}(3)$ is also the map-induced one

$$\Delta(L^{q'}) = R(\Delta(E^q_{ij})) \tag{30}$$

where $\Delta(E^q_{ij})$ is the coproduct of $U_q(3)$ given by (26). However, in this case the corepresentations resulted from $\Delta(E^q_{ij})$ should also be symmetric, otherwise the resulting corepresentations cannot be specified in terms of $U_q(3) \supset SO_{q'}(3) \supset SO(2)$ basis. The map-induced coproduct discussed is not well defined when q is a root of unity [13].

We have shown that, under some conditions, the deforming functionals enable us to construct quantum algebras in Lie algebra chains and vice versa. Especially, some quantum algebra chains can also survive from this technique, e.g. $U_q(3) \supset SO_{q'}(3) \supset SO(2)$ chain. We hope these new realizations can be applied to many physical problems. An investigation of q-deformed IBM for nuclei along this line is also in progress.

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Note added in proof. After completion of this work, the author became aware of [14], in which another form of the q-deformation of the classical $SU(3) \supset SO(3)$ for symmetric $SU(3)$ representations is obtained.

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